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A GENERAL BRANCH AND CUT PROCEDURE FOR STOCHASTIC INTEGER PROGRAMS

by

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ABSTRACT

In this paper, a general branch and cut procedure for stochastic integer programs is presented. It is shown to provide a finite exact algorithm for a large class of stochastic integer programs, even in the presence of binary variables or continuous random variables in the second stage.

Key words : Stochastic integer programming, branch and cut.

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RÉSUMÉ

On décrit dans cet article une procédure générale de "branch and cut" pour la résolution de programmes stochastiques en nombres entiers. La méthode proposée constitue un algorithme exact fini, applicable à une classe générale de programmes stochastiques en nombres entiers, même lorsque ceux-ci comportent des variables binaires ou des variables aléatoires continues en seconde période.

Mots-clés : Programmation stochastique en nombres entiers, "branch and cut".

Introduction

Stochastic integer programs have a well-deserved reputation for being computationally intractable. In fact, they combine two types of programs which by themselves are often difficult to solve. Stochastic programs are modeled in two stages. In the first stage, a decision has to be taken without full information on some random variable. When the value taken by the random variable is known, corrective or recourse actions can be taken in the second stage. Stochastic integer programs are those for which some variables are restricted to be integer, either in the first stage or in the second stage.

To the best of our knowledge, the first proposed solution method was a cutting planes algorithm by Wollmer (1980) for the case when second stage decision variables are continuous and the random variable is discrete. In a similar vein, Louveaux (1986) has shown that multistage programs having the block-separability property are equivalent to two-stage problems of the type solved by Wollmer. Other authors have studied the properties of the expected value of the second stage program when it involves integer decision variables. General results concerning the continuity of this function have been obtained by Stougie (1985). This type of research has also been done for specific applications : e.g. Jaillet (1988) or Laporte and Louveaux (1990) for the analysis of some vehicle routing problems. Several studies have addressed the behaviour of various heuristics for specific problems. These include proofs of asymptotic optimality of some heuristics, based on the study of probabilistic analysis of algorithms. References for these are Rinnooy Kan and Stougie (1988) and Stougie (1985), for example. Some problems can also be solved using Dynamic Programming (see Lageweg *et al.* (1988)).

The object of this paper is to show that a general branch and cut procedure, first proposed by Laporte, Louveaux and Mercure (1989b) in the context of a particular vehicle routing problem, in fact provides an exact algorithm for a large class of stochastic integer problems. Of particular interest is the fact that it can be used for problems also involving binary decision variables in the second stage, and that is also applicable to some problems

involving integer decision variables and continuous random variables.

The structure of the paper is as follows. In Section 1, we review the definition of two-stage stochastic linear programs with recourse. We also provide a simple taxonomy for the various types of stochastic integer programs. We then present in Section 2 one example of an application, the stochastic vehicle routing problem (VRP), which helps to illustrate the proposed method and its applicability. The general branch and bound procedure is presented in Section 3, with definitions needed to prove its finite convergence. Section 4 contains the full study of the application of the branch and cut procedure to the various classes of stochastic integer programs. Examples using the VRP application are given. The conclusion follows in Section 5.

Note that we have chosen to illustrate the applicability of the proposed method to various cases of the vehicle routing problem. This choice of using a single type of application is made solely for the sake of unity and conciseness in the presentation. It certainly does not imply that this is the only type of problem to which the proposed method can be applied. Several other examples in location problems, covering problems or general integer programs could be provided, to name only a few.

1. Two-stage Stochastic Linear Programs with Recourse

The general formulation of a two-stage stochastic linear program with fixed recourse is to find

$$\min z = cx + E_\xi \min \{q(\omega)\cdot y \mid Wy = h(\omega) - T(\omega)\cdot x,\ y \in Y\} \qquad (1)$$

$$\text{s.t.} \quad Ax = b$$

$$x \in X,$$

where c is a known vector in $\Re^{n_1}$, b a known vector in R^{m_1}, ξ a random N-vector defined on the probability space $(\Omega, \mathcal{A}, P)$, and A and W are known matrices of size $m_1 \times n_1$ and $m_2 \times n_2$, respectively. For each ω, $T(\omega)$, $q(\omega)$ and $h(\omega)$ have conformable sizes. Piecing together the stochastic components of the problem, we obtain a vector $\xi(\omega) = (q(\omega)\ h(\omega), T_1(\omega), \ldots, T_{m_2}(\omega))$ with $N = n_2 + m_2 + (m_2 \times n_1)$ components, where $T_i(\omega)$ is the i^{th} row of $T(\omega)$. Transposes have been eliminated for simplicity. E_ξ represents the mathematical expectation with respect to ξ. Let also $\Xi \subseteq \Re^N$ be the support of ξ, i.e. the smallest closed subset in $\Re^N$ such that $P(\xi \in \Xi) = 1$. If ξ has finite support, there exists a discrete probability distribution which describes ξ.

The above problem is equivalent to the so-called deterministic equivalent program :

$$\min \quad z = cx + Q(x) \qquad (2)$$

$$\text{s.t.} \quad Ax = b,\ x \in X$$

where $$Q(x) = E_\xi\, Q(x, \xi(\omega)) \qquad (3)$$

and $$Q(x, \xi(\omega)) = \min_y \{q(\omega)\cdot y \mid Wy = h(\omega) - T(\omega)\cdot x,\ y \in Y\}. \qquad (4)$$

The function $Q(x)$ is called the second stage value function. A comprehensive treatment of stochastic programs can be found in Ermoliev and Wets (1988). Of particular interest in this paper are the cases when the sets X or Y define restrictions requiring some variables to be integer.

In order to characterize the various possible situations, we will use the notation a|b|c to represent a particular class of two-stage programs. The first letter characterizes the first

stage variables, and will be either C if all first stage variables are continuous, B if they are all binary or M when some first stage variables are continuous and other are binary (mixed integer problems). The second letter will similarly characterize the second stage variables with an identical representation. The third letter will be either D when ξ has a finite support (discrete distribution) or C when ξ is a continuous random variable. Thus, C|C|D represents a two-stage continuous stochastic program with a discrete distribution. For this case, well known characterization theorems and solution algorithms exist (see, for example, Wets (1983)). For some classes of problems, it will also be useful to know whether or not they possess the block-separable property (see Louveaux 1986) and this will be denoted by an additional |s. For example, M|M|D|s would be a stochastic program with mixed integer first and second stage, discrete distribution and block-separability. For simplicity, no distinction is made between integer and binary variables, since integer variables can be transformed into binary variables.

2. The Stochastic Vehicle Routing Problem

In this section, we present the vehicle routing problem (VRP) and describe some of its stochastic versions. These examples will serve to illustrate the methods proposed in the next sections.

The vehicle routing problem can be defined as follows. Let $G = (V,E)$ be a graph where $V = \{v_0, v_1, \ldots, v_n\}$ is the vertex set and $E = \{(v_i, v_j) : v_i, v_j \in V,\ i \neq j\}$ is the arc set. Vertex v_0 represents a depot at which are based a maximum of m vehicles and $v_1, \ldots, v_n$ correspond to cities or customers. With E is associated a distance or travel cost matrix $C = (c_{ij})$. If C is symmetrical, as will be the case in this paper, E can be defined as a set of undirected edges (v_i, v_j) where $i < j$. The VRP consists of determining a set of m vehicle routes

(i) starting and ending at the depot ;

(ii) so that every vertex at $V \setminus \{v_0\}$ is visited exactly once by exactly one vehicle ;

(iii) some side constraints on the vehicle routes are satisfied ;

(iv) a linear combination of the number of vehicles and of the total routing cost is minimized.

There exists a wide variety of side constraints. For examples and recent surveys on the VRP, see Laporte and Nobert (1987) and Golden and Assad (1988). There are also several elements that may be stochastic. This is the case, for example, when customers are visited on a random basis (Jaillet (1988)), when vehicles of fixed capacities must collect random quantities at the various customers (see, for example, Laporte, Louveaux and Mercure (1989a)), or when travel times are random (Laporte, Louveaux and Mercure (1989b)).

In all these cases, we assume that first stage decisions consist of choosing the number of vehicles as well as designing planned routes, while second stage decisions will take care of possible corrective or recourse actions.

All these problems can be represented in the following way. Let x_{ij}, $i < j$ denote the number of times edge (v_i, v_j) is used in the optimal solution. This variable is equal to 0 or 1 if $0 < i < j$, and to 0, 1 or 2 if $i = 0$; the case $x_{0j} = 2$ corresponds to a return trip between the depot and v_j. Let f be the fixed cost (per planning period) of a vehicle.

The model is then

$$\min \quad fm + \sum_{i<j} c_{ij} x_{ij} + Q(x) \tag{5}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{0j} = 2m \tag{6}$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad (v_k \in V\backslash\{v_0\}) \tag{7}$$

Illegal route elimination constraints (8)

$$x_{ij} \in \{0, 1\} \qquad (v_i, v_j \in V\backslash\{v_0\}) \tag{9}$$

$$x_{0j} \in \{0, 1, 2\} \qquad (v_j \in V\backslash\{v_0\}) \tag{10}$$

All constraints except (8) are self-explanatory. Illegal routes referred to in constraints (8) include subtours disconnected from the depot. As in the travelling salesman problem (TSP), Dantzig, Fulkerson and Johnson (1954) illegal subtours can be eliminated by imposing the connectivity constraints

$$\sum_{i \in S,\, j \notin S} x_{ij} \geq 2 \qquad (S \subseteq V\backslash\{v_0\},\ 3 \leq |S| \leq n-2) \tag{11}$$

or

$$\sum_{i \notin S,\, j \in S} x_{ij} \geq 2$$

Illegal routes may also include vehicle routes whose total duration or total amount collected exceed a preset value. In a stochastic model however, it is preferred to specifically

consider the recourse actions needed to cope with such failures. The expected cost of these actions is represented by Q(x).

For example, when a vehicle of given capacity D collects random demands, failures can occur along a given route when the vehicle is unable to load a customer's demand. In this case, one possible policy is for the vehicle to return to the depot, unload at the depot and resume its trip where failure occured. The cost of such a recourse action is the cost of the return trip (assuming no penalty for possible overtime). The function Q(x) represents the expected cost of these failures for a given set of tours defined by x. Details for computing Q(x) can be found in Laporte and Louveaux (1990). In the case of random visits to customers, the function Q(x) represents the expected savings (a negative value) for not visiting some customers. The computation of these savings can be found in Jaillet (1988). Mathematical properties of these function Q(x) will be studied in the next sections. Most examples of the stochastic VRP are of the B|B|* type.

3. A General Branch and Cut Scheme

As we shall see in the next section, a variety of stochastic integer programs can be solved using the following branch and cut scheme.

Let $\bar{X}$ be the polytope defined by the set of constraints in X, other than those defining the first stage variables type. Hence $X = \bar{X} \cap \mathfrak{R}^{n_1}_+$ for C|*|* problems, $X = \bar{X} \cap Z^{n_1}_+$ for B|*|* problems, and similarly for M|*|* problems. In this section, we assume that at least some first stage variables are binary. For simplicity, we assume all variables to be non-negative.

At a given stage of the branch and cut process, we will consider the so-called current problem :

$$(CP) \quad \min \quad cx + \Theta \tag{12}$$

$$\text{s.t.} \quad Ax = b \tag{13}$$

$$D_k.x \geq d_k \qquad (k = 1,\ldots,s) \tag{14}$$

$$E_k.x + \Theta \geq e_k \qquad (k = 1,\ldots,t) \tag{15}$$

$$x \geq 0, \ \Theta \in \mathfrak{R}. \tag{16}$$

In this current problem, constraints (14) are used to give an appropriate representation of $\bar{X}$. Usually, $\bar{X}$ is relaxed at the initial stage, and additional constraints (14) are added when needed. These constraints (14) will be referred to as feasibility cuts. A set of feasibility cuts is said to be valid if there exists some finite value s such that $x \in \bar{X}$ if and only if $(D_k.x \geq d_k, \ k = 1,\ldots,s)$. Examples of such relaxed constraints are induced constraints for second stage feasibility or subtour elimination constraints of type (11) in vehicle routing problems. Constraints (11) form a valid set of feasibility cuts to remove all illegal subtours.

Variable Θ, along with the constraints (15), are used to express Θ as an appropriate approximation of Q(x). Constraints

(15) will be referred to as optimality cuts. A set of t optimality cuts (15) is said to be valid if for all $x \in X$, $(x,\Theta) \in \{(x,\Theta) | E_k.x + \Theta \geq e_k,\ k = 1,\ldots,t\}$ implies $\Theta \geq Q(x)$.

Such a valid set of optimality cuts is known to exist for two-stage stochastic linear programs in continuous decision variables with discrete random variables. In the L-shaped method (Van Slyke and Wets (1969)), they are defined using the optimal simplex multipliers of the second stage recourse problem. In practice, it is not required to have $\Theta \geq Q(x)$ for all feasible (x,Θ), but simply for the optimal (x^*,Θ^*).

The purpose of this paper is to show that such a set of valid optimality cuts also exists for a variety of two-stage stochastic integer programs.

We now present the branch and cut procedure.

Step 0 : $s = t = \nu = 0$, $\bar{z} = \infty$; the only pendant node corresponds to the initial current problem.

Step 1 : Select some pendant node in the list ; if none exists, stop.

Step 2 : Set $\nu = \nu + 1$; solve the current problem. If the current problem has no feasible solution, fathom the current node ; go to Step 1. Otherwise, let (x^ν,Θ^ν) be an optimal solution. (If $t = 0$, Θ is set equal to $-\infty$ or to an appropriate lower bound and is ignored in the computation).

Step 3 : Check for any relaxed constraint violation. If one exists, add one feasibility cut (14), set $s = s + 1$ and return to Step 2. If $cx^\nu + \Theta x^\nu > \bar{z}$, fathom the current problem and return to Step 1.

Step 4 : Check for integrality restrictions. If one is violated, create two new branches following the usual branch and cut procedure ; append the new nodes to the list of pendant nodes ; return to Step 1.

Step 5 : Compute $Q(x^v)$ and $z^v = cx^v + Q(x^v)$. If $z^v < \bar{z}$, then update $\bar{z}$.

Step 6 : If $\theta^v \geq Q(x^v)$, then fathom the current node and return to Step 1. Otherwise impose optimality cuts (15) and return to Step 2.

As can be seen, the fathoming rules are different from the usual ones. In particular, nodes are not necessarily fathomed when integrality conditions are satisfied. The following proposition follows immediately from the fathoming rules and from the definitions of valid sets of cuts.

Proposition 1 : For any problem for which a valid set of feasibility cuts and a valid set of optimality cuts exist, the above branch and cut procedure yields an optimal solution (when one exists) in a finite number of steps.

4. Solvable Cases of Stochastic Integer Programs

In this section, we study a number of solvable cases of stochastic integer problems. Note that these cases do not form a partition of stochastic integer problems.

4.1. *|C|D

When the second stage involves continuous variables only and a discrete distribution for ξ, the following result holds.

Proposition 2 : Assume the second-stage variables are continuous. Then $Q(x,\xi)$ is a piecewise linear and convex function of x, and so is $Q(x)$ if ξ has a finite support.

This result justifies the use of outer-linearization techniques, in two stage stochastic programs, as is done in the L-shaped method (Van Slyke and Wets (1969) in the continuous case). When first stage variables are discrete, two techniques can be used.

a. Cutting-Planes Techniques

They combine outer-linearization for approximating $Q(x)$ with cutting planes to recover integrality. This combination is described by Wollmer (1980) who applied it to a class of investment problems.

b. Branch and Cut Techniques

For several families of deterministic integer problems, branch and bound techniques are known to perform better than cutting planes. We can therefore expect the same to hold for stochastic integer problems.

For *|C|D problems, branch and cut techniques combine outer-linearization of $Q(x)$ with branch and bound to recover integrality of first stage variables. The rules to be followed have been explained in Section 3.

In general, constraints (15) imposed in Step 6 of the algorithm can be defined as in the L-shaped method by using expectations based on optimal multipliers of the second stage problem (see Van Slyke and Wets (1969)). In some applications, they can also be defined in a specific manner. Such is the case for instance in routing problems with stochastic travel time and penalties proportional to overtime (see Laporte, Louveaux and Mercure (1989b)).

4.2. *|M|D|s

We recall here a result by Louveaux (1986).

Proposition 3 : A multistage stochastic program with block-separable recourse is equivalent to a two-stage stochastic program where the first stage is the extensive form of the aggregate level problem and the value function of the second stage for one realization of the random vector in the second stage is the sum (weighted by the appropriate probabilities) of the detailed level recourse functions for that realization of the random vector and all its successors.

It follows that, when binary variables are associated with the aggregate level, problems of the class *|M|D|s are equivalent to problems *|C|D. Techniques presented in Section 4.1. are therefore applicable. Of particular interest is the fact that this property also holds for multistage programs.

4.3. B|*|*

We consider here stochastic programs whose first stage decision variables are binary and whose second stage decision variables are binary or mixed integer.

Assumption : Q(x) is bounded below by a finite lower bound L.

Since the first stage decision variables are binary, there are only a finite number of feasible first stage solutions. Let $r = 1,\ldots, R$ index these feasible solutions.

Proposition 4 : Let $x_i = 1$, $i \in S$ and $x_i = 0$, $i \notin S$ be the r^{th} feasible solution and θ_r the corresponding expected second-stage value. Define the optimality cut as

$$\theta \geq (\theta_r - L)\left(\sum_{i \in S} x_i - \sum_{i \notin S} x_i\right) - (\theta_r - L)(|S| - 1) + L, \quad (18)$$

where $|S|$ is the cardinality of S. Then, the set of optimality cuts (18) defined for all first stage feasible solutions is a valid set of optimality cuts.

Proof : The quantity $\sum_{i \in S} x_i - \sum_{i \notin S} x_i$ is always less than or equal to $|S|$. It takes the value $|S|$ only when x is the r^{th} feasible solution. Now, when $\sum_{i \in S} x_i - \sum_{i \notin S} x_i$ is equal to $|S|$, the right-hand side takes the value θ_r and otherwise the right-hand side takes a value less than or equal to L. A set of valid cuts is therefore obtained by imposing one such constraint for each first-stage feasible solution.

□

Given the above proposition, the branch and cut procedure of Section 3 can thus be applied to any B|*|* stochastic program for which a finite lower bound on Q(x) exists and the second stage value function Q(x) can be computed at any feasible first stage decision. This represents a very large class of problems.

The above result is particularly useful for B|B|* problems when the methods of Section 4.1. do not apply.

We now illustrate the application of the above proposition on two specific examples of stochastic VRPs.

The first case is where the demand of customer v_j is 1 with probability p_j and 0 with probability $1 - p_j$, and only those customers with non-zero demands are visited. In the first stage,

routes are planned for all customers. The function Q(x) measures the expected savings generated by not visiting customers with zero demand in the second stage. Its computation can be derived from the work of Jaillet (1988) on the probabilistic traveling salesman problem. When C satisfies the triangle inequality, a lower bound on Q(x) is given by

Proposition 5 : Let $a_j = \min_{\substack{l \neq j \\ k \neq j \\ k \neq l}} \{c_{kl} - c_{jk} - c_{lj}\}$. Then, $L = \sum_{j=1}^{n} a_j p_j$

is a valid lower bound on Q(x).

Proof : Let $q = 1,\ldots,m$ denote the indices of the vehicle routes of the first stage solution x, and let $(v_0, v_{i_1}, \ldots, v_{i_g}, v_{i_{g+1}} \equiv v_0)$ be the q^{th} route, where for simplicity, the dependence of g on q is not made explicit. The first stage cost of that route is equal to its planned length and the expected second stage cost is equal to minus the savings generated by not visiting some customers. For any individual customer v_j $(j \geq 1)$, the saving realized by skipping v_j and v_j only is $-(c_{i_{j-1} i_{j+1}} - c_{i_{j-1} i_j} - c_{i_j i_{j+1}})$. The value of Q(x) is bounded below by

$$\sum_{q=1}^{m} \sum_{j=1}^{g} (c_{i_{j-1} i_{j+1}} - c_{i_{j-1} i_j} - c_{i_j i_{j+1}}) p_j ,$$

since skipping several successive customers never generate a larger saving than the sum of savings associated with the individual customers. The conclusion follows since every customer belongs to exactly one route and each term in parentheses is bounded below by a_j.

□

In the second example, we consider a stochastic VRP with limited vehicle capacity D and random demands. In the first stage, routes are planned for all customers. The expected second stage value function Q(x) represents the expected cost of failure. A failure occurs when the vehicle is unable to load the

customer's demand and the cost of failure is the cost of the return trip from the client to the depot. For this problem, the computation of Q(x) is provided in Laporte, Louveaux and Mercure (1989a). A valid lower bound on Q(x) can be constructed in cases where C satisfies the triangle inequality.

Proposition 6 : Let the customers be ranked in order of increasing distance from the depot : v_{i_1} is the customer closest to the depot, v_{i_2} is the customer next closest to the depot, etc. Let also $p_j = P\left[\sum_{i=1}^{n} d_i > (m + j - 1)\, D\right]$, where d_i represents the demand of customer v_i. Assume $p_j > 0$ for $j = 1 \ldots J$. Then Q(x) is bounded below by

$$L = \sum_{j=1}^{J} p_j \, (c_{i_j 0} + c_{0 i_j}) \qquad (19)$$

Proof : First observe that p_j is a lower bound on the probability of having j failures in total. The validity of the bounds rests on the fact that in (19) the largest failure probabilities are associated with the smallest penalties.

□

A final important remark is in order. The applicability of this branch and cut procedure is not limited to problems with a discrete distribution. In the stochastic VRP with random demands, both the function Q(x) and the lower bound L given in (19) can be computed for continuous random variables, as long as the probability of $\sum_{i \in S} d_i$ exceeding some level can be computed, which is the case for several common random variables.

4.4. C|M|D and M|M|D

As a consequence of the above discussion, the most difficult cases seem to be C|M|D and M|M|D, when the second stage does not possess the block-separability property. When the random variables are discrete, it is possible to transform the two-stage programs as follows : let $y^\xi = (y_B^\xi,\ y_C^\xi)$ be divided into its binary variables y_B^ξ and continuous variables and partition q^ξ and W accordingly. The original program (1) can then be transformed into :

$$\begin{aligned} \min \quad & z = cx + E_\xi\ q_B^\xi\ y_B^\xi + E_\xi\ Q(x,\ y_B^\xi,\ \xi) \\ & Ax = b \\ & x \in X,\ y_B^\xi \in Y_B^\xi, \end{aligned} \tag{20}$$

where

$$Q(x, y_B^\xi, \xi) = \min \{q_C^\xi\ .\ y_C^\xi \mid W_C\ .\ y_C^\xi = h(\omega) - T(\omega)x - W_B\ .\ y_B^\xi,\ y_C^\xi \in Y_C^\xi\}.$$

Program (20) is now of the class M|C|D with an extended first stage problem. When the number of realizations of the random vector ξ remains sufficiently low, such a program is still solvable with the techniques of Section 4.1. An example of such an application can be found in Bienstock and Shapiro (1988).

5. Conclusion

In this paper, we have proposed a characterization of stochastic integer programs. It has been shown that for a large variety of such problems, a general branch and cut procedure can be usefully applied. Its efficiency has been observed in experiments for related problems. Some research effort is still needed to identify, if possible, more efficient sets of valid optimality cuts. Finally, little is known about efficient solutions of M|M|* type problems which do not possess the block-separability property.

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